

The Killing tensors on an n -dimensional manifold with $SL(n, \mathbb{R})$ -structure

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Abstract

In this paper we solve the problem of finding integrals of equations determining the Killing tensors on an n -dimensional differentiable manifold M endowed with an equiaffine $SL(n, \mathbb{R})$ -structure and discuss possible applications of obtained results in Riemannian geometry.

Keywords: differentiable manifold, $SL(n, \mathbb{R})$ -structure, Killing tensors.

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§1. Introduction

1.1. The "structural point of view" of affine differential geometry was introduced by K. Nomizu in 1982 in a lecture at Münster University with the title "What is Affine Differential Geometry?" (see [No 1]). In the opinion of K. Nomizu, the geometry of a manifold M endowed with an *equiaffine structure* is called affine differential geometry.

In recent years, there has been a new wave of papers devoted to affine differential geometry. Today the number of publications (including monographs) on affine differential geometry reached a considerable level. The main part of these publications is devoted to geometry of hypersurfaces (see [Sch] and [SSV] for the history and references).

1.2. In the present paper we solve the problem of finding integrals of equations determining the Killing tensors (see [KSCH] for the definitions, properties and applications) on an n -dimensional differentiable manifold M endowed with an equiaffine structure. The paper is a direct continuation of [St 2]. The same notations are used here.

The first of two present theorems proved in our paper is an affine analog of the statement published in the paper [St 1], which appeared in the process of solving problems in General relativity.

§2. Definitions and results

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2.1. In order to clarify the approach to problem of finding integrals of equations determining the Killing tensors on an n -dimensional differentiable manifold M we shall start with a brief introduction to the subject which emphasizes the notion of an equiaffine $SL(n, \mathbb{R})$ -structure.

Let M be a connected differentiable manifold of dimension n ($n > 2$), and let $L(M)$ be the corresponding bundle of linear frames with structural group $GL(n, \mathbb{R})$. We define $SL(n, \mathbb{R})$ -structure on M as a principal $SL(n, \mathbb{R})$ -subbundle of $L(M)$. It is well known that an $SL(n, \mathbb{R})$ -structure is simply a volume element on M , i.e. an n -form η that does not vanishing anywhere (see [Ko], Chapter I, §2).

We recall the famous problem of the existence of a uniquely determined linear connection ∇ reducible to G for each given G -structure on M (see [C], p. 213). For example, if M is a manifold with a pseudo-Riemannian metric g of an arbitrary index k , then the bundle $L(M)$ admits a unique linear connection ∇ without torsion that is reducible to $O(m, k)$ -structure. Such a connection is called the *Levi-Civita connection*. It is characterized by the following condition $\nabla g = 0$.

A linear connection ∇ having zero torsion and reducible to $SL(n, \mathbb{R})$ is said to be *equiaffine* and can be characterized by the following equivalent conditions (see [Sch], p 99; [SSV], pp. 57-58):

$$(1) \nabla \eta = 0;$$

(2) the Ricci tensor Ric of ∇ is symmetric; that means $Ric(X, Y) = Ric(Y, X)$ for any vector fields $X, Y \in C^\infty TM$.

An *equiaffine* $SL(n, \mathbb{R})$ -structure or an *equiaffine structure* on an n -dimensional differentiable manifold M is a pair (η, ∇) , where ∇ is a linear connection with zero torsion and η is a volume element which is parallel relative to ∇ (see [No 2], p. 43).

The curvature tensor R of an equiaffine connection ∇ admits a point-wise $SL(n, \mathbb{R})$ -invariant decomposition of the form $R = (n - 1)^{-1}[id_M \otimes Ric - Ric \otimes id_M] + W$ where W is the *Weyl projective curvature tensor* (see [SSV], p. 73-74; [E], §40). Then two classes of equiaffine structures can be distinguished in accordance with this decomposition: the *Ricci-flat* equiaffine $SL(n, \mathbb{R})$ -structures for which $Ric = 0$, and the *equiprojective* $SL(n, \mathbb{R})$ -structures for which

$$R = (n - 1)^{-1}[id_M \otimes Ric - Ric \otimes id_M]. \quad (2.1)$$

Remark 1. Recall that a linear connection ∇ with zero torsion is called *Ricci-flat* if the Ricci tensor $Ric = 0$ (see [M]). On the anther hand, a connection ∇ is called *equiprojective* if

the Weyl projective curvature tensor $W = 0$ (see [Sch], §18). In the literature equiprojective connections sometimes are called *projectively flat* (see, for example, [SSV], p. 73).

An autodiffeomorphism of the manifold M is an automorphism of $SL(n, \mathbb{R})$ -structure if and only if it preserves the volume element η . Let X be a vector field on M . The function $\text{div}X$ defined by the formula $(\text{div}X)\eta = L_X\eta$ where L_X is the Lie differentiation in the direction of the vector field X is called the divergence of X with respect to the n -form η (see [KN], Appendix no. 6). Obviously, X is an infinitesimal automorphism of an $SL(n, \mathbb{R})$ -structure if and only if $\text{div}X = 0$. Such a vector field X is said to be *solenoidal*.

For an arbitrary vector field X on M with a linear connection ∇ we can introduce the tensor field $A_X = L_X - \nabla_X$ regarded as a field of linear endomorphisms of the tangent bundle TM . If M is an n -dimensional with an equiaffine $SL(n, \mathbb{R})$ -structure then the formula $\text{trace}A_X = -\text{div}X$ can be verified directly (see [KN], Appendix no. 6).

We have the $SL(n, \mathbb{R})$ -invariant decomposition $A_X = -n^{-1}(\text{div}X)\text{id}_M + \dot{A}_X$ at every point $x \in M$.

Two classes of vector fields on M endowed with an equiaffine $SL(n, \mathbb{R})$ -structure can be distinguished in accordance with this decomposition: the *solenoidal vector fields* and the *concircular vector fields* for which, by definition (see [Sc], p. 322; [M]), we have $A_X = -n^{-1}(\text{div}X)\text{id}_M$.

The integrability conditions of the structure equation $A_X = -n^{-1}(\text{div}X)\text{id}_M$ of the concircular vector field X is the Ricci's identity $Y(\text{div}X)Z - Z(\text{div}X)Y = nR(Y, Z)X$ for any vector fields $Y, Z \in C^\infty TM$ (see [E], §11). This identity are equivalent to the condition $W(Y, Z)X = 0$ for any vector fields $Y, Z \in C^\infty TM$. It follows that an equiaffine $SL(n, \mathbb{R})$ -structure on an n -dimensional manifold M is equiprojective if and only if there exist n linearly independent concircular vector fields X_1, X_2, \dots, X_p on M (see also [ST]). This statement is an affine analog of the well known fact for the Riemannian manifold M of constant sectional curvature (see [F]).

Remark 2. A pseudo-Riemannian manifold (M, g) with a projectively flat Levi-Civita connection ∇ is a manifold of constant section curvature (see [Sch], §18). Therefore a manifold M endowed with an equiprojective $SL(n, \mathbb{R})$ -structure is an affine analog of a pseudo-Riemannian manifold of constant sectional curvature.

2.2. We consider an n -dimensional manifold M with an equiaffine $SL(n, \mathbb{R})$ -structure and denote by $\Lambda^p M$ ($1 \leq p \leq n-1$) the p^{th} exterior power $\Lambda^p(T^*M)$ of the cotangent bundle T^*M of M . Hence $C^\infty \Lambda^p M$, the space of all C^∞ -sections of $\Lambda^p M$, is the space of skew-symmetric covariant tensor fields of degree p ($1 \leq p \leq n-1$).

Let $\gamma : J \subset \mathbb{R} \rightarrow M$ be an arbitrary geodesic on M with affine parameter $t \in J$. In this case, we have $\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = 0$ for the tangent vector $\frac{d\gamma}{dt}$ of γ .

Definition 1 (see [St 2]). A skew-symmetric tensor field $\omega \in C^\infty \Lambda^p M$ ($1 \leq p \leq n-1$) on an n -dimensional manifold M with an equiaffine $SL(n, \mathbb{R})$ -structure is called Killing-Yano tensor of degree p if the tensor $i_{\frac{d\gamma}{dt}} \omega := \text{trace} \left(\frac{d\gamma}{dt} \otimes \omega \right)$ is parallel along an arbitrary geodesic γ on M .

From this definition we conclude that $\left(\nabla_{\frac{d\gamma}{dt}} \omega \right) \left(\frac{d\gamma}{dt}, X_2, \dots, X_p \right) = 0$ for any vector fields $X_2, \dots, X_p \in C^\infty TM$. Since the geodesic γ may be chosen arbitrary, the above relation is possible if and only if $\nabla \omega \in C^\infty \Lambda^{p+1} M$, which is equivalent to $d\omega = (n+1)\nabla \omega$ for the exterior differential operator $d : C^\infty \Lambda^p M \rightarrow C^\infty \Lambda^{p+1} M$.

Obviously, the set of Killing-Yano tensors of degree p ($1 \leq p \leq n-1$) constitutes an \mathbb{R} -module of tensor fields on M , denoted by $\mathbf{K}^p(M, \mathbb{R})$.

Let X_1, \dots, X_p be p linearly independent concircular vector fields on M ($1 \leq p \leq n-1$). Then direct inspection shows that the tensor field ω of degree $n-p$ dual to the tensor field $\text{alt}\{X_1 \otimes \dots \otimes X_p\}$ relative to the n -form η is a Killing-Yano tensor (see also [St 2]). Therefore on any n -manifold M with equiprojective $SL(n, \mathbb{R})$ -structure, there exist at least $n![p!(n-p)!]^{-1}$ linearly independent Killing-Yano tensors (see [St 2]). Moreover the following theorem is true.

Theorem 1. On an n -dimensional manifold M endowed with an equiprojective $SL(n, \mathbb{R})$ -structure (η, ∇) , there exist a local coordinate system x^1, \dots, x^n in which an arbitrary Killing-Yano tensor ω of degree p ($1 \leq p \leq n-1$) has the components

$$\omega_{i_1 \dots i_p} = e^{(p+1)\psi} (A_{i_0 i_1 \dots i_p} x^{i_0} + B_{i_1 \dots i_p}) \quad (2.1)$$

where $A_{i_0 i_1 \dots i_p}$ and $B_{i_1 \dots i_p}$ are arbitrary constants skew-symmetric w.r.t. all their indices and $\psi = (n+1)^{-1} \ln(\eta)$.

From the theorem we conclude that the maximum of linearly independent the Killing-Yano tensors is by calculating the number K_n^p of independent $A_{i_0 i_1 \dots i_p}$ and $B_{i_1 \dots i_p}$ which exist after accounting for the symmetries on the indices. It follows that $K_n^p = \frac{(n+1)!}{(p+1)!(n-p)!}$ is the maximum number linearly independent the Killing-Yano tensors.

Corollary 1. Let M be an n -dimensional manifold endowed with an equiprojective $SL(n, \mathbb{R})$ -structure then

$$\dim K^p(M, \mathbb{R}) = \frac{(n+1)!}{(p+1)!(n-p)!}.$$

On our fixed manifold M with an equiaffine $SL(n, \mathbf{R})$ -structure, we denote by $S^p M$ the bundle of symmetric covariant tensor fields of degree p on M . Hence $C^\infty S^p M$, the space of all C^∞ -sections of $S^p M$, is the space symmetric covariant tensor fields of degree p .

Definition 2 (see [St 2]). *A symmetric tensor field $\varphi \in C^\infty S^p M$ on an n -dimensional manifold M with an equiaffine $SL(n, \mathbf{R})$ -structure is called Killing tensor of degree p if $\varphi \left(\frac{d\gamma}{dt}, \dots, \frac{d\gamma}{dt} \right) = \text{const}$ along an arbitrary geodesic γ on M .*

Let $\varphi \left(\frac{d\gamma}{dt}, \dots, \frac{d\gamma}{dt} \right) = \text{const}$ along an arbitrary geodesic γ on M and hence φ is a Killing tensor. Then the above relation is possible if and only if $\delta^* \varphi := \sum_{cicl} \{ \nabla \varphi \} = 0$ where for the local components $\nabla_{i_0} \varphi_{i_1 \dots i_p}$ of $\nabla \varphi$ we define the sum $\sum_{cicl} \{ \nabla_{i_0} \varphi_{i_1 \dots i_p} \}$ as the sum of the terms obtained by a cyclic permutation of the indices i_0, i_1, \dots, i_p .

Obviously, the set of Killing tensors constitutes an \mathbf{R} -module of tensor fields on M , denoted by $\mathbf{T}^p(M, \mathbf{R})$.

Let M be an n -dimensional manifold endowed with an equiprojective $SL(n, \mathbf{R})$ -structure (η, ∇) , and $\omega_1, \dots, \omega_p$ be p linearly independent Killing-Yano tensors of degree 1 on M . Then direct inspection shows that the tensor field $\varphi := \text{sym}\{\omega_1 \otimes \dots \otimes \omega_p\}$ is a Killing tensor of degree p . Therefore on any n -manifold M with equiprojective $SL(n, \mathbf{R})$ -structure, there exist at least $(n + p - 1)! [p!(n - 1)!]^{-1}$ linearly independent Killing tensors (see also [SS]). Moreover the following theorem is true.

Theorem 2. *On an n -dimensional manifold M endowed with an equiprojective $SL(n, \mathbf{R})$ -structure (η, ∇) , there exist a local coordinate system x^1, \dots, x^n in which the components $\varphi_{i_1 \dots i_p}$ of an arbitrary Killing tensor φ of degree p can be expressed in the form of an p^{th} degree polynomial in the x^i 's*

$$\varphi_{i_1 \dots i_p} = e^{2p\psi} \sum_{q=0}^p A_{i_1 \dots i_p j_1 \dots j_q} x^{j_1} \dots x^{j_q} \quad (2.2)$$

where the coefficients $A_{i_1 \dots i_p j_1 \dots j_q}$ are constant and symmetric in the set of indices i_1, \dots, i_p and the set of indices j_1, \dots, j_q . In addition to these properties the coefficients $A_{i_1 \dots i_p j_1 \dots j_q}$ have the following symmetries

$$\sum_{cicl} \{ A_{i_1 \dots i_p j_1 \dots j_{p-s}} \}_{j_{p-s+1}} = 0 \quad (2.3)$$

for $s = 1, \dots, p - 1$ and

$$\sum_{cicl} \{ A_{i_1 \dots i_p j_1} \} = 0. \quad (2.4)$$

From the theorem we conclude that the maximum number of linearly independent the Killing

tensors is obtained by calculating the number T_n^p of independent $A_{i_1 \dots i_p j_1 \dots j_q}$ ($q = 0, 1, \dots, n$) which exist after accounting for the symmetries on the indices the dependence relations (2.3) and (2.4). By [KL] it follows that

$$T_n^p = \frac{p(p+1)^2(p+2)^2 \dots (m+p-1)^2(m+p)}{(p+1)!p!}$$

is the maximum number linearly independent the Killing-Yano tensors. Then we have the following proposition.

Corollary 2. *Let M be an n -dimensional manifold endowed with an equiprojective $SL(n, \mathbf{R})$ -structure then*

$$\dim T^p(M, \mathbf{R}) = \frac{p(p+1)^2(p+2)^2 \dots (m+p-1)^2(m+p)}{p!(p+1)!}.$$

§3. Proofs of theorems

3.1. We let $f : \bar{M} \rightarrow M$ denote the mapping of an \bar{n} -dimensional manifold \bar{M} endowed with an equiaffine $SL(\bar{n}, \mathbf{R})$ -structure onto another an n -dimensional manifold M endowed with an equiaffine $SL(n, \mathbf{R})$ -structure, and let f_* be the differential of this mapping. For any covariant tensor field ω on M , we can then define the covariant tensor field $f^*\omega$ on \bar{M} , where f^* is the transformation transposed to the transformation f_* .

If $\dim \bar{M} = \dim M = n$ and $f : \bar{M} \rightarrow M$ is a projective diffeomorphism, i.e., a mapping that transforms an arbitrary geodesic in \bar{M} into a geodesic in M , then we have the following lemma.

Lemma 1. *Let $f : \bar{M} \rightarrow M$ be a projective diffeomorphism of n -dimensional manifolds endowed with the equiaffine $SL(n, \mathbf{R})$ -structures $(\bar{\eta}, \bar{\nabla})$ and (η, ∇) respectively. Then for an arbitrary Killing-Yano tensor ω of degree p ($1 \leq p \leq n-1$) on the manifold M the tensor field $\bar{\omega} = e^{-(p+1)\psi}(f^*\omega)$ with $\psi = (n+1)^{-1} \ln(\eta/\bar{\eta})$ will be the Killing-Yano tensor of degree p on the manifold \bar{M} .*

Proof. It is known that the diffeomorphism $f : \bar{M} \rightarrow M$ can be realized following the principle of equality of the local coordinates $\bar{x}^1 = x^1, \dots, \bar{x}^n = x^n$ at the corresponding points \bar{x} and $x = f(\bar{x})$ of these manifolds. In this case, we have the equalities (see [Sch], §18; [M] and [YI])

$$\Gamma_{ij}^k = \bar{\Gamma}_{ij}^k + \psi_i \delta_j^k + \psi_j \delta_i^k \quad (3.1)$$

for the objects Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$ of the a equiaffine connections ∇ and $\bar{\nabla}$ in the coordinate system x^1, \dots, x^n that is common w. r. t. the mapping $f : \bar{M} \rightarrow M$, and for the gradient $\psi_j = (n+1)^{-1} \partial_j \ln[\eta/\bar{\eta}]$.

Equalities (3.1) imply that the mapping f^{-1} , which is inverse to the projective diffeomorphism $f : \bar{M} \rightarrow M$, is a projective mapping.

We set $\omega_{i_1 \dots i_p}$ be the local components of a Killing-Yano tensor ω of degree p ($1 \leq p \leq n-1$) arbitrary defined on the manifold M ; by definition, these components satisfy the equations

$$\nabla_{i_0} \omega_{i_1 \dots i_p} + \nabla_{i_1} \omega_{i_0 \dots i_p} = 0. \quad (3.2)$$

From equalities (3.2) we find directly that the components

$$\bar{\omega}_{i_1 \dots i_p} = e^{-(p+1)\psi} \omega_{i_1 \dots i_p} \quad (3.3)$$

of the tensor field $\bar{\omega} = e^{-(p+1)\psi} (f^* \omega)$ satisfy the equations

$$\bar{\nabla}_{i_0} \bar{\omega}_{i_1 \dots i_p} + \bar{\nabla}_{i_1} \bar{\omega}_{i_0 \dots i_p} = 0. \quad (3.4)$$

Hence, the tensor field $\bar{\omega}$ is a Killing-Yano tensor of degree p ($1 \leq p \leq n-1$) on the manifold \bar{M} .

3.2. Let \mathbf{A}^n be an n -dimensional affine space with a volume element given by the determinant: $\det(e_1, \dots, e_n) = 1$, where $\{e_1, \dots, e_n\}$ is the standard basis of the underlying vector space for \mathbf{A}^n . We denote by ∇ the standard linear connection in \mathbf{A}^n relative to which the volume element "det" is parallel (see [No 2]; [SSV], p. 10).

Let $f : \bar{M} \rightarrow \mathbf{A}^n$ be a projective diffeomorphism from a manifold \bar{M} endowed with equiaffine $SL(n, \mathbb{R})$ -structure onto an affine space \mathbf{A}^n endowed with standard equiaffine $SL(n, \mathbb{R})$ -structure. It is well known that manifolds endowed with equiprojective $SL(n, \mathbb{R})$ -structures and only these manifolds are projectively diffeomorphic to an affine space \mathbf{A}^n (see [Sch], §18; [M]) therefore in our case the $SL(n, \mathbb{R})$ -structure of the manifold \bar{M} must be an equiprojective structure.

If \mathbf{A}^n is an affine space with the Cartesian system of coordinates $\bar{x}_1, \dots, \bar{x}^n$ then the components $\bar{\omega}_{i_1 \dots i_p}$ of the Killing-Yano tensor $\bar{\omega}$ of degree p ($(1 \leq p \leq n-1)$) in equation (3.4) must now satisfy

$$\partial_j \bar{\omega}_{ii_1 \dots i_p} + \partial_i \bar{\omega}_{ji_1 \dots i_p} = 0 \quad (3.5)$$

where $\partial_j = \frac{\partial}{\partial x^j}$. From (3.5) we conclude the following equations

$$\partial_k \partial_j \bar{\omega}_{ii_1 \dots i_p} + \partial_k \partial_i \bar{\omega}_{ji_1 \dots i_p} = 0; \quad (3.6)$$

$$\partial_j \partial_i \bar{\omega}_{ki_1 \dots i_p} + \partial_j \partial_k \bar{\omega}_{ii_1 \dots i_p} = 0; \quad (3.7)$$

$$\partial_i \partial_k \bar{\omega}_{ji_1 \dots i_p} + \partial_i \partial_j \bar{\omega}_{ki_1 \dots i_p} = 0. \quad (3.8)$$

From (3.6), (3.7), (3.8) we find

$$\partial_k \partial_j \bar{\omega}_{i_1 i_2 \dots i_p} = 0, \quad (3.9)$$

by using identities $\frac{\partial^2 h}{\partial \bar{x}^k \partial \bar{x}^j} = \frac{\partial^2 h}{\partial \bar{x}^j \partial \bar{x}^k}$ which are carried out for an arbitrary smooth function $h : \mathbf{A}^n \rightarrow \mathbf{R}$. The integrals of equations (3.9) take the form

$$\bar{\omega}_{i_1 \dots i_p} = A_{i_0 i_1 \dots i_p} \bar{x}^{i_0} + B_{i_1 \dots i_p} \quad (3.10)$$

for any skew-symmetric constants $A_{i_0 i_1 \dots i_p}$ and $B_{i_1 \dots i_p}$ (see also [SS] and [St 3]). Taking the components (3.10) of the Killing-Yano tensor $\bar{\omega}$ in \mathbf{A}^n and using Lemma 1, we can formulate Theorem 1.

3.3. Let \bar{M} be a manifold of dimension n endowed with the equiaffine $SL(n, \mathbf{R})$ -structure $(\bar{\eta}, \bar{\nabla})$ and M be a manifold of some dimension endowed with the equiaffine $SL(n, \mathbf{R})$ -structure (η, ∇) . Let there is given a projective diffeomorphism $f : \bar{M} \rightarrow M$, then we have the following lemma.

Lemma 2. *Let $f : \bar{M} \rightarrow M$ be a projective diffeomorphism of n -dimensional manifolds endowed with the equiaffine $SL(n, \mathbf{R})$ -structures $(\bar{\eta}, \bar{\nabla})$ and (η, ∇) respectively. Then for an arbitrary Killing tensor φ of degree p on the manifold M the tensor field $\bar{\varphi} = e^{-2p\psi}(f^*\varphi)$ with $\psi = (n+1)^{-1} \ln(\eta/\bar{\eta})$ will be the Killing tensor of degree p on the manifold \bar{M} .*

Proof. We set $\varphi_{i_1 \dots i_p}$ to be components of the Killing tensor φ arbitrary defined on the manifold M ; by definition, these components satisfy the following equations $\sum_{cicl} \{\nabla_{i_0} \varphi_{i_1 \dots i_p}\} = 0$. Then we find directly that the components $\bar{\varphi}_{i_1 \dots i_p} = e^{-2p\psi} \varphi_{i_1 \dots i_p}$ of the tensor $\bar{\varphi} = e^{-2p\psi} \varphi$ satisfy the equations

$$\sum_{cicl} \{\bar{\nabla}_{i_0} \bar{\varphi}_{i_1 \dots i_p}\} = e^{-2p\psi} \sum_{cicl} \{\nabla_{i_0} \varphi_{i_1 \dots i_p}\} = 0. \quad (3.11)$$

From (3.11) we conclude that the tensor field $\bar{\varphi}$ is a Killing tensor of degree p on the manifold \bar{M} .

3.4. It follows from Nijenhuis (see [Ni]) that in an n -dimensional affine space \mathbf{A}^n the components $\bar{\varphi}_{i_1 \dots i_p}$ of the Killing tensor $\bar{\varphi}$ of degree p can be expressed in the form of an p^{th} degree polynomial in the \bar{x}^i 's

$$\varphi_{i_1 \dots i_p} = e^{-2p\psi} \sum_{q=0}^p A_{i_1 \dots i_p j_1 \dots j_q} \bar{x}^{j_1} \dots \bar{x}^{j_q}. \quad (3.12)$$

The coefficients $A_{i_1 \dots i_p j_1 \dots j_q}$ are constant and symmetric in the set of indices i_1, \dots, i_p and the set of indices j_1, \dots, j_q . In addition to these properties the coefficients $A_{i_1 \dots i_p j_1 \dots j_q}$ have the following symmetries $\sum_{cycl} \{A_{i_1 \dots i_p j_1 \dots j_{p-s}}\}_{j_{p-s+1}} = 0$ for $s = 1, \dots, p-1$ and $\sum_{cycl} \{A_{i_1 \dots i_p j_1} = 0\}$. Taking the components (3.12) of the Killing tensor $\bar{\varphi}$ in \mathbf{A}^n and using Lemma 2, we can formulate Theorem 2.

§4. Applications to Riemannian geometry

4.1. Let (M, g) be a pseudo-Riemannian manifold of dimensional n . Then from the present theorems 1 and 2 we conclude that an arbitrary Killing vector ω has the following local covariant components $\omega_i = e^{2\psi}(A_{ik}x^k + B_i)$ where $\psi = [2(n+1)]^{-1} \ln|\det g|$, A 's and B 's are constants and $A_{ik} + A_{ki} = 0$ (see also [St 1]). It follows that the group of infinitesimal isometric transformations has $\frac{1}{2}n(n+1)$ parameters (see also [E], §71).

4.2. Following [T] and [Ka], a skew-symmetric covariant tensor field ϑ of degree p ($1 \leq p \leq n-1$) is called a conformal Killing tensor if $\vartheta \in \ker D$ for

$$D = \nabla - \frac{1}{p+1}d - \frac{1}{n-p+1}g \wedge d^*$$

where d^* is the codifferential operator $d^* : C^\infty \Lambda^{p+1} M \rightarrow C^\infty \Lambda^p M$ and

$$(g \wedge d^* \vartheta)_{i_0 i_1 \dots i_p} = \sum_{a=1}^p (-1)^{a+1} g_{i_0 i_a} (d^* \vartheta)_{i_1 \dots \hat{i}_a \dots i_p}.$$

Obviously, the set of conformal Killing tensors constitutes an vector space of tensor fields on (M, g) , denoted by $\mathbf{C}^p(M, \mathbb{R})$ (see [St 5]). If a conformal Killing tensor ϑ belongs to $\ker d^*$, then it is a Killing-Yano tensor. On the other hand, if a conformal Killing tensor ϑ belongs to $\ker d$, it is called a closed conformal Killing tensor or a planar tensor (see [St 4] and [St 5]). We denote the vector space of these tensors by $\mathbf{P}^p(M, \mathbb{R})$.

By [Ka] on an arbitrary n -dimensional pseudo-Riemannian manifold (M, g) of constant nonzero sectional curvature C ($C \neq 0$) the vector space $\mathbf{C}^p(M, \mathbb{R})$ of conformal Killing tensors is decomposed uniquely in the form

$$\mathbf{C}^p(M, \mathbb{R}) = \mathbf{K}^p(M, \mathbb{R}) \oplus \mathbf{P}^p(M, \mathbb{R}). \quad (4.1)$$

From (4.1) we conclude that any conformal Killing tensor ϑ of degree p is decomposed uniquely in the form $\vartheta = \omega + \theta$ where ω is a Killing-Yano tensor of degree p and θ is a closed conformal Killing tensor of degree p .

Following theorem 1, on an n -dimensional pseudo-Riemannian manifold (M, g) of constant nonzero sectional curvature C ($C \neq 0$) there is a local coordinate system x^1, \dots, x^n in which an arbitrary Killing-Yano tensor ω of degree p ($2 \leq p \leq n-1$) has the components

$$\omega_{i_1 \dots i_p} = e^{(p+1)\psi} (A_{i_0 i_1 \dots i_p} x^{i_0} + B_{i_1 \dots i_p}) \quad (4.2)$$

where $\psi = [2(n+1)]^{-1} \ln |det g|$, $\psi_k = \frac{\partial \psi}{\partial x^k}$ and $A_{i_0 i_1 \dots i_p}$, $B_{i_1 \dots i_p}$ are arbitrary skew-symmetric constants. On the other hand, by [St 3] on a pseudo-Riemannian manifold (M, g) of constant nonzero curvature C ($C \neq 0$) the components $\theta_{i_1 \dots i_p}$ of a closed conformal Killing tensor θ of degree p ($1 \leq p \leq n-1$) can be found from the equations

$$\theta_{i_1 i_2 \dots i_p} = -\frac{1}{pC} \nabla_{i_1} \omega_{i_2 \dots i_p} \quad (4.3)$$

where $\nabla_{i_1} \omega_{i_2 \dots i_p} = \partial_{i_1} \omega_{i_2 \dots i_p} - \omega_{k \dots i_p} \Gamma_{i_2 i_1}^k - \dots - \omega_{i_2 \dots k} \Gamma_{i_p i_1}^k$ is the expression for the covariant derivative $\nabla \omega$ of the Killing-Yano tensor of degree $p-1$. Moreover, by virtue of (3.1) on a pseudo-Riemannian manifold (M, g) of constant curvature C ($C \neq 0$) the Christoffel symbols Γ_{ij}^k have the following form $\Gamma_{ij}^k = \psi_i \delta_j^k + \psi_j \delta_i^k$ (see also [St 1]). Therefore, we can deduce from (4.2) and (4.3) that

$$\theta_{i_1 \dots i_p} = -\frac{1}{C} e^{p\psi} (\psi_{[i_1} A_{|k| i_2 \dots i_p]} x^k + \psi_{[i_1} B_{i_2 \dots i_p]} + \frac{1}{p} A_{i_1 i_2 \dots i_p}).$$

Consequently we have

Theorem 3. *On an n -dimensional pseudo-Riemannian manifold (M, g) of constant nonzero sectional curvature C ($C \neq 0$) there is a local coordinate system x^1, \dots, x^n in which an arbitrary conformal Killing tensor ϑ of degree p ($2 \leq p \leq n-1$) has the components*

$$\vartheta_{i_1 \dots i_p} = e^{(p+1)\psi} (A_{k i_1 \dots i_p} x^k + B_{i_1 \dots i_p}) - \frac{1}{C} e^{p\psi} (\psi_{[i_1} C_{|k| i_2 \dots i_p]} x^k + \psi_{[i_1} D_{i_2 \dots i_p]} + \frac{1}{p} C_{i_1 i_2 \dots i_p})$$

where $\psi = [2(n+1)]^{-1} \ln |det g|$, $\psi_k = \frac{\partial \psi}{\partial x^k}$ and $A_{i_0 i_1 \dots i_p}$, $B_{i_1 \dots i_p}$, $C_{i_1 \dots i_p}$ and $D_{i_1 \dots i_p}$ are arbitrary skew-symmetric constants.

Remark 3. For a conformal Killing vector field, see K. Yano and T. Nagano [YN].

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